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Static and dynamic analyses of tensegrity structures. Part II. Quasi-static analysis

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Abstract

Linearized Lagrangian equations developed in the first part of the paper were employed for static analyses of cyclic cylindrical tensegrity modules. Linearized equilibrium equations at natural configurations were used to investigate initial shape, static and kinematic indeterminancy, pre-stress and infinitesimal mechanism modes, and the sensitivity analysis of initial geometry. Linearized equilibrium equations at pre-stressed initial configurations were utilized to investigate pre-stress stiffening and to distinguish first-order mechanisms from higher-order mechanisms. To estimate critical loads for bar buckling and cable slacking, nonlinear equilibrium equations were employed to compute element forces. Further, the equivalence between the twist angle theorem obtained from a geometrical consideration and the equilibrium analysis was established for cyclic cylindrical tensegrity modules. It is concluded that infinitesimal mechanism modes and pre-stresses characterize the static and dynamic response of tensegrity structures. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In order to examine static and kinematic determinancy of structures, small deformation equilibrium equations have been investigated. For a truss structure with n_E elements or members and n_N nodes or joints with n_C displacement constraints, there are $n_V \equiv 3n_N - n_C$ unknown displacement components. In this paper, it is assumed that $n_C \geqslant 6$ to constrain each structure against rigid body motion. Let the element internal force vector be denoted by $\mathbf{s}$, an $n_E \times 1$ column matrix, and the external nodal force vector by $\mathbf{f}$, an $n_V \times 1$ column matrix. The initial equilibrium equation for small deformation is expressed as

$$\mathbf{A}\mathbf{s} = \mathbf{f}, \tag{1}$$

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where **A** is an $n_V \times n_E$ matrix which depends on the initial geometry through direction cosines of truss elements (1.24b,c). In this second part of the paper, equations from the first part are referred to with the preface of 1, such as (1.24b,c) for (24b,c) of the first part.

Maxwell (1864, 1890) classified the stiffness of a truss by the difference between the element number n_E and the active or unknown degrees of freedom n_V:

$$\mathrm{Mx} \equiv n_E - n_V, \tag{2}$$

where Mx is the Maxwell number. According to Maxwell, if $\mathrm{Mx} > 0$, truss structures are redundant. If $\mathrm{Mx} = 0$, trusses may be statically determinate. If $\mathrm{Mx} < 0$, trusses become kinematically indeterminate. As exceptions to the above for $\mathrm{Mx} \leqslant 0$, Maxwell noted structures which exhibit "inferior order of stiffness", i.e., the stiffness of the order of pre-stress. Calladine (1978) observed that most of tensegrity structures introduced by Mark and Fuller (1973) possess mechanisms that could be stiffened by pre-stress to achieve "infinitesimal mechanisms" which exhibit Maxwell's inferior-order stiffness. By investigating the vector spaces associated with **A** in Eq. (1), Calladine (1978) obtained the relationship between the number of pre-stress modes n_S and the number of infinitesimal mechanism modes n_M as

$$\mathrm{Mx} = n_S - n_M. \tag{3}$$

Both Maxwell and Calladine utilized static equilibrium equations.

From a view of kinematic constraints of rigid linkages, infinitesimal mechanisms of truss structures were also investigated. A representative cross-section of this approach may be found in the papers by Kuznetsov (1988, 1991a,b) who investigated the constraints of element lengths in the Pfaffian form: $\mathbf{As} = \mathbf{0}$ and derived Eq. (3).

By analytically establishing the equilibrium equations of space trusses with cyclic cylindrical symmetry, Tarnai (1980a) obtained the tensegrity conditions with infinitesimal mechanisms. In a subsequent paper, Tarnai (1980b) classified mechanisms into "finite" and "infinitesimal" mechanisms. The former is a mechanism with finite motion, while the latter is that with infinitesimal motion with the stiffness of inferior order. Tarnai posed two questions regarding statically and kinematically indeterminate structures: (i) a pre-stressability condition and (ii) a criterion for distinguishing infinitesimal mechanisms. Koiter (1984) rephrased Tarnai's question to distinguish first-order infinitesimal mechanisms from higher-order infinitesimal mechanisms. Pellegrino and Calladine (1986) successfully answered Tarnai's questions by computing linearly independent row and column vectors of **A** for (i) and by counting the dimensions of the "product" forces for (ii).

In traditional truss (frame) design, designers place nodes freely to meet their design objectives by properly connecting them to form a simple or compound space truss to satisfy rank $\mathbf{A} = n_E$ (e.g., Timoshenko and Young, 1965; McCormac, 1984). On the contrary, in the tensegrity structural design with first-order infinitesimal mechanisms, i.e., rank $\mathbf{A} < n_E$, designers cannot place nodes arbitrarily. Typically, a designer selects a specific tensegrity "module" with a specific pre-stressed mode **s** and assembles the modules to form a desired structure. In tensegrity structural design and analysis, initial configurations must be found. This process is referred to as "shape finding" (Hanaor, 1988) or "form finding" (Motro, 1990). Furthermore, tensegrity structures experience large deformations due to infinitesimal mechanisms.

The objective of the second part of the paper is to perform equilibrium analyses of basic tensegrity modules by utilizing linearized Lagrangian equations. The following results were obtained from static analyses:

(i) Pre-stress and infinitesimal mechanism modes were defined based upon a singular value decomposition of the initial equilibrium matrix **A**.
(ii) The existence of maximum or minimum length elements was shown for cyclic cylindrical tensegrity modules by finding the equivalence between Tarnai's solution (1980a) and the twist angle theorem presented by Tobie (1967) and Kenner (1976).

(iii) Starting from a known tensegrity structure, a constraint equation was presented for shape modification in the configuration space. Further, the pre-stress and infinitesimal mechanism modes are insensitive to initial geometrical imperfection and small changes of geometry.
(iv) An analytical expression was obtained for the stiffness of pre-stressed tensegrity structures. If the tangent stiffness matrix is positive definite, the structure is a first-order infinitesimal mechanism. Otherwise, it is a higher-order infinitesimal mechanism;
(v) Element forces were computed to estimate critical loads for bifurcation-type slacking of cables and buckling of bars.

Due to page limitations, only basic tensegrity modules were investigated as examples. Analyses of spherical tensegrity modules and tensegrity structures with repeated tensegrity modules are deferred to subsequent publications. In the following, linearized truss equations are employed to investigate: (i) initial equilibrium, (ii) initial shape finding of cyclic cylindrical tensegrity modules, (iii) stiffness of pre-stressed tensegrity structures, and (iv) sensitivity of initial geometrical imperfection. Initial equilibrium analyses reveal that a class of tensegrity structures with $\mathrm{Mx} \leqslant 0$ is both statically and kinematically indeterminate.

2. Initial equilibrium analysis

For small deformation of a truss structure at the natural state, Eq. (1) is derived from Eqs. (1.24b) and (1.24c) by noting that $\mathbf{g}^{(e)} = \mathbf{G}^{(e)}$ at $t = 0$. Eq. (1) is also obtained from (1.35) at $t = 0$ with (1.34c) evaluated at $\mathbf{d} = \mathbf{0}$. The present derivation of pre-stress and infinitesimal mechanism modes differs from that of Pellegrino and Calladine (1986) in the use of: (i) Clapeyron's theorem instead of the principle of virtual work and (ii) the singular value decomposition of $\mathbf{A}$ instead of the row-echelon form of the augmented $\mathbf{A}$.

Let $\mathbf{R}^{n_V}$ denote the vector space of nodal displacement vectors $\mathbf{d}$ of $n_V \times 1$ column matrices, and $\mathbf{R}^{n_E}$ signify the vector space of element elongation vectors $\boldsymbol{\varepsilon}$ of $n_E \times 1$ column matrices. The linear work functionals $\mathbf{f}^*$ form a co-vector space $\mathbf{R}^{*n_V}$. The corresponding nodal force vector $\mathbf{f}$ in $\mathbf{R}^{n_V}$ is defined via the inner product as

$$\mathbf{f}^*(\mathbf{d}) \equiv \langle \mathbf{f}, \mathbf{d} \rangle_{n_V} \equiv \mathbf{f} \cdot \mathbf{d}. \tag{4a}$$

Similarly, the strain energy functionals $\mathbf{s}^*$ form a co-vector space $\mathbf{R}^{*n_E}$. The element internal force vector $\mathbf{s}$ in $\mathbf{R}^{n_E}$ is defined via the inner product as

$$\mathbf{s}^*(\boldsymbol{\varepsilon}) \equiv \langle \mathbf{s}, \boldsymbol{\varepsilon} \rangle_{n_E} \equiv \mathbf{s} \cdot \boldsymbol{\varepsilon}. \tag{4b}$$

The equilibrium equation (1) defines a linear transformation $\mathbf{A}$ from $\mathbf{R}^{*n_E}$ to $\mathbf{R}^{*n_V}$. For small deformation, Clapeyron's theorem (Sokolnikoff, 1956) states that the work done by surface traction and body forces acting through the displacements from the natural state to the deformed equilibrium configuration is equal to twice the strain energy of the body if it obeys Hooke's law. The theorem yields

$$\langle \mathbf{f}, \mathbf{d} \rangle_{n_V} = \langle \mathbf{s}, \boldsymbol{\varepsilon} \rangle_{n_E}. \tag{5}$$

By substituting the linear transformation (1) into Eq. (5), one obtains the adjoint (transpose) transformation of $\mathbf{A}$ from $\mathbf{R}^{n_V}$ to $\mathbf{R}^{n_E}$ as

$$\langle \mathbf{A}\mathbf{s}, \mathbf{d} \rangle_{n_V} \equiv \langle \mathbf{s}, \mathbf{A}^{\mathrm{T}}\mathbf{d} \rangle_{n_E}. \tag{6}$$

This definition of the element elongation ε agrees, as it should, with that obtained from Eqs. (1.32a) and (1.32c) evaluated at $\mathbf{d} = \mathbf{0}$ for small deformations:

$$\boldsymbol{\varepsilon} = \mathbf{A}^{\mathrm{T}}\mathbf{d}. \tag{7}$$

The null and range spaces of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ are defined as

$$\text{null } \mathbf{A} \equiv \{\mathbf{s} \,|\, \mathbf{A}\mathbf{s} = \mathbf{0}, \mathbf{s} \in \mathbf{R}^{n_E}\}, \tag{8a}$$

$$\text{null } \mathbf{A}^T \equiv \{\mathbf{d} \,|\, \mathbf{A}^T\mathbf{d} = \mathbf{0}, \mathbf{d} \in \mathbf{R}^{n_V}\}, \tag{8b}$$

$$\text{range } \mathbf{A} \equiv \{\mathbf{f} \,|\, \mathbf{f} = \mathbf{A}\mathbf{s}, \forall \mathbf{s} \in \mathbf{R}^{n_E}\}, \tag{8c}$$

$$\text{range } \mathbf{A}^T \equiv \{\boldsymbol{\varepsilon} \,|\, \boldsymbol{\varepsilon} = \mathbf{A}^T\mathbf{d}, \forall \mathbf{d} \in \mathbf{R}^{n_V}\}. \tag{8d}$$

Let dim ( ) denote the dimension of ( ). By noting that $r_A \equiv \dim \mathbf{A} = \dim \mathbf{A}^T = \dim(\text{range } \mathbf{A}) = \dim(\text{range } \mathbf{A}^T)$, $r_A \leqslant n_E$, and $r_A \leqslant n_V$, Calladine (1978) observed

$$n_E = n_S + r_A, \tag{9a}$$

$$n_V = n_M + r_A, \tag{9b}$$

where $n_S \equiv \dim(\text{null } \mathbf{A})$ is the number of pre-stress modes, and $n_M \equiv \dim(\text{null } \mathbf{A}^T)$ is the number of infinitesimal mechanism modes. From Eqs. (2), (9a) and (9b), Calladine's relation (3) was derived. Further, Pellegrino and Calladine (1986) presented a physical interpretation of Eqs. (8a)–(8d). Structures are statically indeterminate if $n_S \geqslant 1$ and kinematically indeterminate if $n_M \geqslant 1$.

Referring to Fig. 1, null **A** includes pre-stressed element force modes in self-equilibrium; null $\mathbf{A}^T$ contains mechanism modes without elongation; range **A** consists of admissible external loads; and range $\mathbf{A}^T$ is composed of elongation modes induced by displacements. After merging the co-vector spaces with vector spaces in Fig. 1 by introducing the inner products (4a) and (4b), the physical interpretation of the complementary spaces becomes available. Let $(\ )^\perp$ denote the orthogonal complement of ( ). The subspace $(\text{range } \mathbf{A}^T)^\perp = \text{null } \mathbf{A}$ consists of incompatible element elongation vectors; $(\text{range } \mathbf{A})^\perp = \text{null } \mathbf{A}^T$ contains inadmissible external loads; $(\text{null } \mathbf{A}^T)^\perp = \text{range } \mathbf{A}$ is composed of displacements with nonzero element elongation; and $(\text{null } \mathbf{A})^\perp = \text{range } \mathbf{A}^T$ includes element force modes in equilibrium with admissible external loads.

To obtain base vectors in the above spaces, the singular value decomposition theorem (Noble and Daniel, 1977) is utilized. Pellegrino (1993) also employed this theorem for kinematic and static analyses of equilibrium matrices. It is known that $n \times n$ symmetric matrices have real eigenvalues and orthonormal sets of n eigenvectors. Therefore, $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ have, respectively, n_E and n_V eigenpairs.

Further, both $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are positive semi-definite, i.e., $\det(\mathbf{A}^T\mathbf{A}) \geqslant 0$ and $\det(\mathbf{A}\mathbf{A}^T) \geqslant 0$ indicating that their eigenvalues are positive or zero. There are $r_A (\equiv \text{rank } \mathbf{A} = \text{rank } \mathbf{A}^T)$ positive eigenvalues: $\sigma_1^2 \geqslant \sigma_2^2 \geqslant \cdots \geqslant \sigma_{r_A}^2 > 0$ where positive σ's are called the singular values of **A**. The ordered eigenpairs in the decreasing singular values satisfy

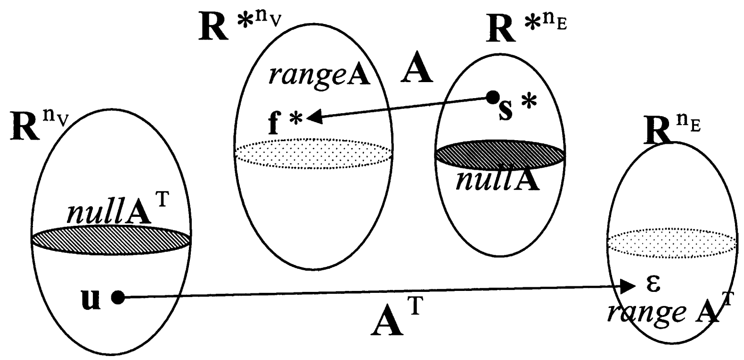

Fig. 1. Null and range spaces of **A** and $\mathbf{A}^T$ in the spaces of nodal force/displacement and element force/elongation vectors.

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{s}_i = \sigma_i^2\mathbf{s}_i, \quad i = 1, 2, \ldots, n_{\mathrm{E}}, \tag{10a}$$

$$\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{d}_j = \sigma_j^2\mathbf{d}_j, \quad j = 1, 2, \ldots, n_{\mathrm{V}}. \tag{10b}$$

In $\mathbf{R}^{n_{\mathrm{E}}}$, $\widetilde{\mathbf{S}} \equiv [\mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_{n_{\mathrm{E}}}]$ is an orthonormal basis, while $\widetilde{\mathbf{D}} \equiv [\mathbf{d}_1\ \mathbf{d}_2\ \ldots\ \mathbf{d}_{n_{\mathrm{V}}}]$ is an orthonormal basis in $\mathbf{R}^{n_{\mathrm{V}}}$. By using the $n_{\mathrm{V}} \times n_{\mathrm{V}}$ orthonormal matrix $\widetilde{\mathbf{D}}$ and the $n_{\mathrm{E}} \times n_{\mathrm{E}}$ orthonormal matrix $\widetilde{\mathbf{S}}$, $\mathbf{A}$ is decomposed into

$$\mathbf{A} = \widetilde{\mathbf{D}}\widetilde{\Sigma}\widetilde{\mathbf{S}}, \tag{11a}$$

where $\widetilde{\Sigma}$ is an $n_{\mathrm{V}} \times n_{\mathrm{E}}$ matrix with singular values on the diagonals of the first r_{A} rows:

$$\widetilde{\Sigma} \equiv \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 & 0 \\ 0 & \sigma_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \sigma_{r_{\mathrm{A}}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{11b}$$

The eigenvectors $\mathbf{s}_i$ corresponding to zero eigenvalues of Eq. (10a) span null $\mathbf{A}$ composed of pre-stress modes, while eigenvectors $\mathbf{d}_j$ corresponding to zero eigenvalues of Eq. (10b) span null $\mathbf{A}^{\mathrm{T}}$ consisting of mechanism modes. The physical interpretation of Eqs. (8a)–(8d) becomes clear if the following coordinate transformations are made:

$$\hat{\mathbf{s}} = \widetilde{\mathbf{S}}^{\mathrm{T}}\mathbf{s}, \tag{12a}$$

$$\hat{\boldsymbol{\varepsilon}} = \widetilde{\mathbf{S}}\boldsymbol{\varepsilon}, \tag{12b}$$

$$\hat{\mathbf{f}} = \widetilde{\mathbf{D}}^{\mathrm{T}}\mathbf{f}, \tag{12c}$$

$$\hat{\mathbf{d}} = \widetilde{\mathbf{D}}^{\mathrm{T}}\mathbf{d}. \tag{12d}$$

Eqs. (1) and (7) are rewritten as

$$\widetilde{\Sigma}\hat{\mathbf{s}} = \hat{\mathbf{f}}, \quad \widetilde{\Sigma}^{\mathrm{T}}\hat{\mathbf{d}} = \hat{\boldsymbol{\varepsilon}}. \tag{13}$$

In the first term of Eq. (13), the $\hat{\mathbf{s}}$-components corresponding to zero eigenvalues of Eq. (10a) represent the amplitudes of pre-stress modes, while the $\hat{\mathbf{f}}$-components corresponding to zero on the left-hand side denote the amplitudes of inadmissible external loads. In the second term of Eq. (13), the $\hat{\mathbf{d}}$-components corresponding to zero eigenvalues of Eq. (10b) describe the amplitudes of infinitesimal mechanism modes, while $\hat{\boldsymbol{\varepsilon}}$-components corresponding to zero on the left-hand side represent the amplitudes of incompatible element elongations.

The eigenproblems of Eqs. (10a) and (10b) can be easily solved by using, for example, Jacobi's method (Bathe, 1982) or the Lanczos method with shifting (Hughes, 1987).

3. Shape finding of cyclic cylindrical modules

In this paper, a class of tensegrity modules which satisfies $\mathrm{Mx} \equiv n_{\mathrm{E}} - n_{\mathrm{V}} \leqslant 0$ and $n_{\mathrm{S}} \equiv \dim(\text{null }\mathbf{A}) = 1$ are considered. For regular trusses with $\mathrm{Mx} = 0$, det $\mathbf{A} > 0$ implies statically and kinematically determinate structures. On the contrary, tensegrity structures collapse if $\det\mathbf{A} > 0$ due to the earth's gravity or other minute disturbances since slender cables cannot carry compression. Tensegrity structures with $\mathrm{Mx} \leqslant 0$ can only exist if there is a pre-stress mode, i.e., $\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = 0$, to stiffen the structures. Further, due to Calladine's relation (3), tensegrity structures with $\mathrm{Mx} \leqslant 0$ possess both infinitesimal mechanism and pre-stress modes. There are some "redundant" tensegrity modules with $\mathrm{Mx} > 0$ and rank $\mathbf{A} = n_{\mathrm{V}}$. These are excluded from the present study since they do not require initial shape finding and do not possess infinitesimal mechanisms.

As basic examples, regular cylindrical tensegrity modules are considered in this section. They have been utilized as basic tensegrity modules to build more complex tensegrity structures such as a two-stage tensegrity (Skelton and Sultan, 1997) and a double-layer tensegrity shell (Hanaor, 1988, 1993). The condition for the existence of a pre-stress mode, $\det\mathbf{A} = 0$ for $\mathrm{Mx} = 0$, i.e., $n_{\mathrm{S}} \equiv \dim(\mathrm{null}\ \mathbf{A}) \geqslant 1$ was analytically investigated by Tarnai (1980a). Since rank $\mathbf{A} = \dim(\mathrm{range}\ \mathbf{A})$ is the number of independent columns of $\mathbf{A}$ defined by element direction cosines, a "brute force" experiment to find dim(null $\mathbf{A}$) can be conducted by cutting a cable of an existing tensegrity module one at a time. If the module collapses, the maximum number of cables cut shows dim(null $\mathbf{A}$). An analytical method is to count the algebraic multiplicity of each real root of the characteristic equation $\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = 0$. Tarnai's characteristic equation, Eq. (12), shows that $\dim(\mathrm{null}\ \mathbf{A}) = 1$ for regular cylindrical tensegrity modules. Unfortunately, it is not easy to build $\mathbf{A}$ and analytically find $\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = 0$.

A simpler alternative approach is possible for some regular cylindrical and spherical tensegrity modules. Both Möbius (1837) and Maxwell (1890) noted that a tensegrity state renders one or more elements to their maximum or minimum lengths. The twist angle theorem, which was credited to Tobie (1967) by Kenner (1976), yields Tarnai's tensegrity conditions from a simple geometrical analysis. In the following, the twist angle theorem is introduced and the equivalence between the geometrical approach and the equilibrium approach is established.

Consider regular cylindrical tensegrity modules inscribed by a conical cylinder whose height is bounded by top and base regular n-polygons, $n = 3, 4, 5, \ldots$ Fig. 2 illustrates a set of generating nodes and elements. Let the nodes of the top n-polygon be numbered in the counterclockwise direction from 1 to n, while the corresponding nodes on the base n-polygon be numbered from $(n+1)$ to $2n$. The top n-polygon is twisted by $\alpha + 2\pi/n$ in the counter-clockwise direction, as shown in Fig. 2. Let the height of the cylinder be denoted by h. The radii of the top and base circles circumscribing the n-polygons are r_{h} and r_0, respectively. A special

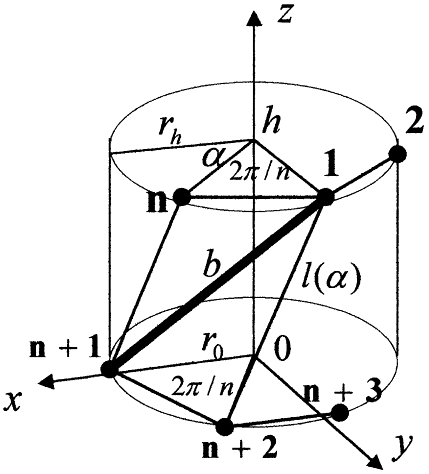

Fig. 2. The twist angle theorem for an n-bar cyclic cylindrical tensegrity module.

case $r_h = r_0$ yields the right cylindrical truss investigated by Tarnai (1980a). A bar element connects nodes 1 and $(n+1)$. The cable elements connect the nodes: n to 1, $(n+1)$ to $(n+2)$, and $(n+2)$ to 1. Let the origin of the Cartesian coordinate system $\{x, y, z\}$ be placed at the center of the base circle with the z-axis in the axis of the cylinder and node $n+1$ on the x-axis. The cylindrical coordinates of the nodes are 1 $(r_h,\ 2\pi/n+\alpha, h)$, n $(r_h,\ \alpha,\ h)$, $n+1$ $(r_0,\ 0,\ 0)$, $n+2$ $(r_0,\ 2\pi/n,\ 0)$. It is assumed that the bar length b and the radii of the circles, r_h and r_0, are fixed. The length of the equilateral cables of the top n-polygon is $l_h = 2r_h\sin(\pi/n)$, while the length of the cables of the base n-polygon is $l_0 = 2r_0\sin(\pi/n)$. Let the length of the vertical cable connecting nodes 1 and $(n+2)$ be l. One computes b and l in terms of r_h, r_0, h, and α as

$$b = \sqrt{r_h^2 + r_0^2 - 2r_h r_0 \cos\left(\frac{2\pi}{n}+\alpha\right) + h^2}, \tag{14a}$$

$$l = \sqrt{r_h^2 + r_0^2 - 2r_h r_0 \cos\alpha + h^2}. \tag{14b}$$

If h is eliminated from Eq. (14b) by using Eq. (14a), one finds

$$l(\alpha) = \sqrt{b^2 + 2r_h r_0\left\{\cos\left(\frac{2\pi}{n}+\alpha\right) - \cos\alpha\right\}}. \tag{15a}$$

The extremization of l with respect to the twist angle α, i.e., $\mathrm{d}l/\mathrm{d}\alpha = 0$, yields

$$\sin\left(\frac{2\pi}{n}+\alpha\right) = \sin\alpha = \sin(\pi-\alpha). \tag{15b}$$

The twist angle theorem of Tobie (1967) and Kenner (1976) is obtained from Eq. (15b) as

$$\alpha = \frac{\pi}{2} - \frac{\pi}{n}. \tag{16}$$

The result shows that the twist angle is independent of the tapering r_h/r_0. One can obtain the same result if the top n-polygon is twisted in the clockwise direction. The twist angle predicted by Eq. (16) agrees with Tarnai's twist condition (13) and (14). The corresponding pre-stress mode can be obtained by solving equilibrium equations (1.17) with $\mathbf{f}_j = \mathbf{0}$, at nodes 1 and $(n+2)$. Let the internal element forces due to the pre-stress be denoted by s_b, s_h, s_0, and s_v, respectively, for the bar, the top cable, the base cable, and the vertical cable. An admissible pre-stress mode is obtained with tension in cables and compression in bars as

$$[\,s_v \quad s_h \quad s_0\,] = -\frac{s_b}{b}[\,l \quad r_0 \quad r_h\,]. \tag{17}$$

All nodal coordinates are determined by $l(\alpha)$ and fixed parameters, n, b, r_h, and r_0. The elements of $\mathbf{A} = [a_{ij}]$ consisting of element direction cosines become functions of the variable length: a_{ij} $(l(\alpha))$. Since $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is a positive semi-definite continuous function of α:

$$Q(\alpha) \equiv \det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) \geqslant 0, \tag{18}$$

$Q(\alpha) = 0$ if and only if $\mathrm{d}Q/\mathrm{d}\alpha = 0$.

$$\frac{\mathrm{d}Q}{\mathrm{d}\alpha} = \left(\sum_{i,j=1}^{n_E}\mathrm{Cof}(c_{ij})\frac{\mathrm{d}c_{ij}}{\mathrm{d}l}\right)\frac{\mathrm{d}l}{\mathrm{d}\alpha} = 0, \tag{19}$$

where $\mathrm{Cof}(c_{ij})$ denotes the cofactor of $c_{ij} \equiv a_{ki}\ a_{kj}$ with summation for $k = 1, \ldots, n_V$. If rank $\mathbf{A} = n_E - 1$, which implies $n_S \equiv \dim(\text{null } \mathbf{A}^{\mathrm{T}}\mathbf{A}) = 1$, there is at least one nonzero cofactor of $\mathbf{C} \equiv \mathbf{A}^{\mathrm{T}}\mathbf{A}$. Therefore,

$\mathrm{d}Q/\mathrm{d}\alpha = 0$ if and only if $\mathrm{d}l/\mathrm{d}\alpha = 0$. By combining the above results, $\mathrm{d}l/\mathrm{d}\alpha = 0$ becomes necessary and sufficient conditions for $Q = 0$. The sign of the second derivative of l with α decides whether l takes a maximum or a minimum at $\mathrm{d}Q/\mathrm{d}\alpha = 0$. This geometrical method has only a limited value since it works only for a class of regular tensegrity modules whose direction cosines are described by one unknown length $l(\alpha)$ of a parameter α. However, when this method works, it is much simpler than the equilibrium approach reported by Tarnai (1980a). The above method could fail and yield an unrealistic configuration at $\mathrm{d}l/\mathrm{d}\alpha = 0$, such as vanishing elements, $l = 0$. Further, if there are more than one unknown lengths, the geometrical method fails.

In summary, the conditions for tensegrity structures are $\mathrm{Mx} \leqslant 0$, $\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = 0$, and the existence of an admissible pre-stress mode with tension in a continuous set of cables and compression in discontinuous bars. It will be shown in the sequel that the application of a pre-stress mode transforms tensegrity structures from kinematically indeterminate mechanisms to kinematically determinate first-order infinitesimal mechanisms. According to Koiter (1984), first-order mechanisms exhibit infinitesimal displacements accompanied by second order elongation of at least some of the elements. The regular cylindrical tensegrity module, shown in Fig. 2, possesses an infinitesimal mechanism characterized by twisting bars in the counter-clockwise direction. For $n = 3$ and 4, the twisting mechanisms computed by using Jacobi's method are shown in Fig. 3a and b, respectively. In the figures, solid lines show infinitesimal mechanism modes and dashed lines show the equilibrium configurations.

In Fig. 2, if one rotates the top n-polygon by α instead of $\alpha + 2\pi/n$ in the counter-clockwise direction, the condition $\mathrm{d}l/\mathrm{d}\alpha = 0$ is only found for even n as

$$\alpha = \frac{\pi}{n}, \quad n = 2, 4, 6, \ldots \tag{20}$$

The same result is obtained if one rotates the top n-polygon by α in the clockwise direction. The two symmetric configurations for $n = 4$ were plotted by Tarnai (1980a) in his Fig. 4c and d. By constructing equilibrium equations at a top node and a base node, and by computing a pre-stress mode, one finds that the pre-stresses alternate signs in adjacent top and base cables. Furthermore, it will be shown in the sequel that due to the alternating sign of pre-stresses, the stress stiffening effect cancels out. As a result, the truss structure becomes a higher-order mechanism. According to Koiter (1984), an infinitesimal mechanism is referred to as second order if there exists an infinitesimal motion such that no element undergoes an elongation lower than the third order.

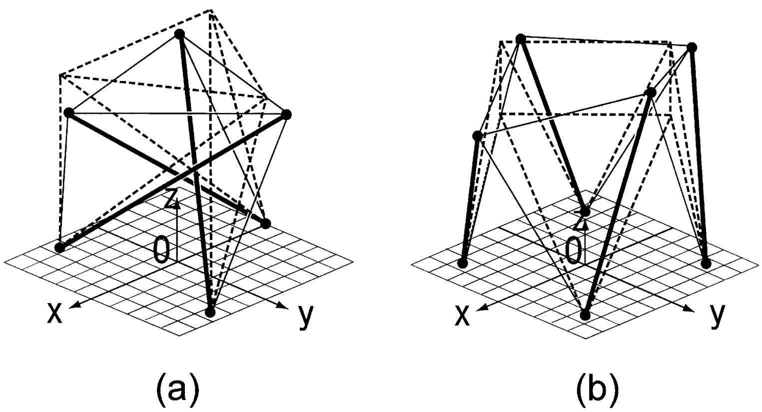

Fig. 3. Infinitesimal mechanism modes of (a) three-bar tensegrity module and (b) four-bar tensegrity module.

4. Stiffness of pre-stressed tensegrity structures

For a pre-stressed tensegrity structure, static equilibrium equations for small deformations are obtained from Eqs. (1.36a)–(1.36c). The tangent stiffness $\mathbf{K}_T$ may be decomposed into the initial stiffness $\mathbf{K}_0$, employed for small deformation truss analyses, and the pre-stress stiffness $\mathbf{K}_s$ as

$$\mathbf{K}_T\mathbf{d} \equiv (\mathbf{K}_0 + \mathbf{K}_s)\mathbf{d} = \mathbf{f}, \tag{21a}$$

where

$$\mathbf{K}_0 \equiv \sum_{e=1}^{n_E} \mathbf{L}g^{(e)^T}\mathbf{K}_0^{(e)}\mathbf{L}g^{(e)}, \tag{21b}$$

$$\mathbf{K}_s \equiv \sum_{e=1}^{n_E} \mathbf{L}g^{(e)^T}\mathbf{K}_s^{(e)}\mathbf{L}g^{(e)}, \tag{21c}$$

$$\mathbf{K}_0^{(e)} \equiv \left(\frac{Y_0A_0}{l_0}\right)^{(e)} \begin{bmatrix} \mathbf{GG}^T & -\mathbf{GG}^T \\ -\mathbf{GG}^T & \mathbf{GG}^T \end{bmatrix}^{(e)}, \tag{21d}$$

$$\mathbf{K}_s^{(e)} \equiv \left(\frac{S_{11}(0)A_0}{l_0}\right)^{(e)} \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{21e}$$

Eqs. (21c) and (21e) show that the pre-stress stiffness at each node is "isotropic" since the nodal stiffness is the same in all directions. If $n_S \equiv \dim(\text{null } \mathbf{A}) = 1$ and tensegrity modules are cyclic or spherical, all elements have nonzero pre-stress $S_{11}(0)$. Therefore, pre-stress stiffening is applied to all elements. Further, for a mechanism mode $\mathbf{d}_M = \mathbf{d}_{n_V}$ in $\widetilde{\mathbf{D}}$ of Eq. (11a), the stiffness is induced only by $\mathbf{K}_s$ since $\mathbf{K}_0\,\mathbf{d}_M = \mathbf{0}$. Pellegrino and Calladine (1986) called $\mathbf{K}_s\,\mathbf{d}_M$ the "product force". They used the number of linearly independent product forces to distinguish first-order mechanisms from higher-order mechanisms. The subsequent development regarding mechanisms may be found in the papers by Kuznetsov (1988), Calladine and Pellegrino (1991, 1992), and Pellegrino (1993).

Since $\mathbf{K}_T$ is available in this paper, det $\mathbf{K}_T$ is utilized for the classification of infinitesimal mechanisms. If det $\mathbf{K}_T > 0$, pre-stressed tensegrity structures are first-order mechanisms, and if det $\mathbf{K}_T = 0$, they are higher-order mechanisms. When the pre-stresses are nonzero in every element and det $\mathbf{K}_T > 0$, it is observed from Eqs. (21c) and (21e) that a single pre-stress mode stiffens "all infinitesimal mechanisms".

It is expected that the initial response of pre-stressed tensegrity structures is characterized by soft infinitesimal mechanism modes resisted only by $\mathbf{K}_s$ in Eq. (21c). Since the infinitesimal mechanism modes are global in regular cylindrical and spherical tensegrity modules, the initial response of tensegrity modules is always "global". Even if one tries to locally deform the tensegrity module shown in Fig. 3a by applying a nodal force at one node, the initial deformation is described by a global twisting of bars.

Examples of second-order mechanisms include Tarnai's symmetric, cyclic cylindrical truss for even n with the top n-polygon rotated by α as in Eq. (20), instead of $\alpha + 2\pi/n$ as illustrated in Fig. 2. This structure does not possess a tensegrity pre-stress mode. Rather, the pre-stress mode induces compression in cables. One can show after straightforward calculations that the diagonal terms of $\mathbf{K}_s$ vanish to cause det $\mathbf{K}_T = 0$. To investigate the response of the structure with higher-order mechanisms, one has to recourse to the nonlinear equilibrium equation (1.23) or (1.35). The updated Lagrangian FE code, which solves Eq. (1.23), was employed for quasi-static loading to confirm that the infinitesimal mechanism was in fact second order.

5. Hardening-type load–displacement relations

In other works, the stability of tensegrity structures was investigated by using different mathematical criteria (Connelly and Whiteley, 1992). Here, the stability analysis is performed within the framework of large deformation kinematics and statics of trusses. The stiffness analysis, presented in the above, is only valid for small deformation at the neighborhood of an initial pre-stressed configuration. Upon loading, infinitesimal mechanisms, illustrated in Fig. 3a, b, are activated and induce large deformation. Fig. 4 illustrates the vertical load and load-point displacement relation of the three-bar cylindrical tensegrity module, shown in Fig. 3a. The same vertical load is applied in the z-direction at each top node. The solid lines are the prediction of the updated-Lagrangian FE code at various initial pre-stress amplitudes. The predictions of the FE code were validated by using Mathematica which also solved nonlinear equilibrium equation (1.23). Model experiments qualitatively agree with the FE prediction. The initial tangent stiffness near the origin of Fig. 4, predicted by Eq. (21a), increases linearly with increasing pre-stress, see Eq. (21e). As Fig. 4 illustrates, the linear range is extremely small. If bars do not buckle, the load–displacement relation reveals increasing stiffness as Eqs. (1.32c) and (1.34c) predict. Skelton and Adhikari (1998) first reported the axial load–displacement relation of a two-stage tensegrity, which also exhibited hardening response. For tensegrity structures with infinitesimal mechanisms, the load–displacement relation is characterized by a nonlinear hard spring.

Both buckling of bars and slacking of cables are of a bifurcation type since an unstable equilibrium path of a straight cable or bar under compression exists. A post-buckling behavior of beams (bars) is "imperfection sensitive" as Budiansky (1974) explained. A critical load should be determined by either conducting the experiments or performing a nonlinear FE analysis. For the loading shown in Fig. 4, the element forces are plotted in Fig. 5 as a function of load-point displacement. The cable tensile forces increased with the vertical loading in both tension and compression, and the cables did not slack. Cable slacking was only observed under loading modes that excite deformation modes without mechanism modes. Cable slacking was reported by Skelton and Adhikari (1998) for the bending deformation of a two-stage tensegrity structure.

Let a load parameter be denoted by λ so that $\mathbf{f} = \lambda \mathbf{f}_0$. The corresponding displacement $\mathbf{d}(\lambda)$ is obtained as a nonlinear function of λ from the quasi-static equilibrium equation (1.23) or (1.35). An estimate for slacking or buckling loads could be obtained by first identifying slacking cables or buckling bars and then

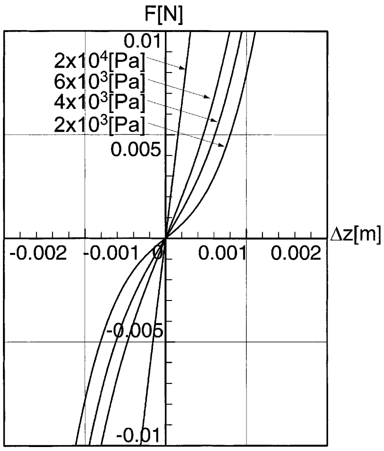

Fig. 4. Load–displacement relation of the three-bar tensegrity module at various pre-stress amplitudes.

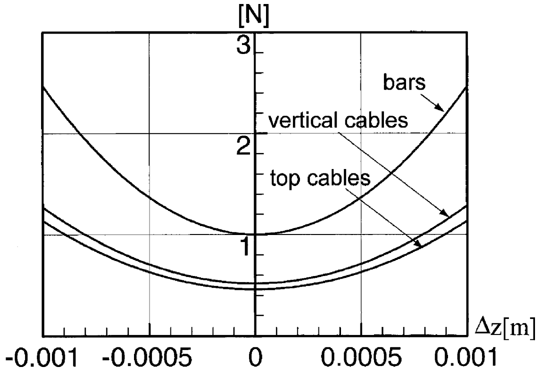

Fig. 5. Element forces–load-point displacement relation of the three-bar tensegrity module with the initial bar pre-stress force 10^4 Pa.

computing the stress increment from Eqs. (1.12a), (1.32a) and (1.32c) with $\mathbf{d}(\lambda)$. In this section, bars and cables are assumed to be linearly elastic with Young's modulus Y_0 independent of E_{11} in Eq. (1.12b). When the sum of the element force increment and the initial element force becomes a critical load $-P_{cr}$, of the element, a buckling or slacking is expected to occur. The element critical values are $P_{cr} = 0$ for cables and a fraction of the Euler critical load: $P_{cr} = \pi^2 Y_0 I / l_0^2 / \mathrm{SF}$ for bars, where I is the areal moment of inertia of the beam (bar) and SF is a safety factor. In the Euler critical load for beam buckling, the safety factor SF = 2.5 or 3 is used to reflect the imperfection sensitivity. If $\mathbf{d}(\lambda)$ is available, a critical load parameter λ_{cr} may be estimated by using (1.12a) and computing the strain increment from Eqs. (1.25a), (1.31a)–(1.31e) as

$$(S_{11}A_0)^{(e)}(0) + \left(\frac{Y_0 A_0}{l_0}\right)^{(e)} \left\{ \begin{bmatrix} -\mathbf{G}^{\mathrm{T}} & \mathbf{G}^{\mathrm{T}} \end{bmatrix}^{(e)} + \frac{1}{l_0^{(e)}} \mathbf{d}^{(e)}(\lambda_{cr})^{\mathrm{T}} \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \right\} \mathbf{d}^{(e)}(\lambda_{cr}) = -P_{cr}, \tag{22a}$$

where

$$\mathbf{d}^{(e)}(\lambda) = \mathbf{L}g^{(e)}\mathbf{d}(\lambda). \tag{22b}$$

Fig. 5 shows the resultant element forces, i.e. the left-hand side of Eq. (22a), computed by using the updated Lagrangian FE code. Several attempts to estimate the second term of Eq. (22a) on the left-hand side by using $\mathbf{d}(\lambda) \approx \lambda\, \mathbf{d}_{\mathrm{M}}$ with a mechanism mode generalized for large rotation failed, yielding a much stiffer response than that shown in Fig. 4. Therefore, to obtain reliable critical loads, nonlinear equilibrium equations must be solved for element forces. In the next section, the sensitivity analysis against initial geometrical imperfection or coordinate changes will be discussed.

6. A tensegrity configuration space

Design of tensegrity structures involves moving nodes of existing structures. Let n_{D} be the number of active (or adjustable) nodal coordinates. For example, the dimension n_{D} of the configuration space of the three-bar tensegrity in Fig. 4a with base nodes fixed is nine. In a configuration or design parameter space $\mathbf{R}^{n_{\mathrm{D}}}$, an existing tensegrity structure is denoted by a vector ξ_0. The constraint equation in the configuration space is

$$Q = \det\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) = 0. \tag{23}$$

The above holonomic constraint (Frankel, 1997) defines a hyper-surface for accessible configurations in $\mathbf{R}^{n_D}$ with $(n_D - 1)$ degrees of freedom if $n_S \equiv \dim(\text{null } \mathbf{A}) = 1$. The normal direction to the hyper-surface is not accessible by satisfying the equilibrium equations with pre-stresses from ξ_0. The admissible direction $\dot{\xi}$ is, therefore, orthogonal to grad Q and is defined by

$$dQ(\dot{\xi}) = \langle \text{grad } Q, \dot{\xi} \rangle_{n_D} = 0. \tag{24a}$$

An admissible coordinate change $\Delta\xi_a$ may be obtained from a trial $\Delta\xi$ by using the Gram–Schmidt procedure with an appropriate vector norm $\| \; \|$ as

$$\Delta\xi_a \approx \Delta\xi - \langle \text{grad } Q, \Delta\xi \rangle \frac{\text{grad } Q}{\|\text{grad } Q\|}. \tag{24b}$$

The forbidden neighborhood configurations can only be realized by applying external nodal forces. Upon the removal of external forces, a pre-stressed tensegrity structure returns to a configuration on the equilibrium hyper-surface.

For an n-bar tensegrity module in Fig. 2, Eq. (14a) with a twist angle defined by Eq. (16) gives a quarter circle with radius $R^2 = b^2 - r_0^2$ in the r_h, h-plane:

$$r_h^2 + h^2 = b^2 - r_0^2. \tag{25}$$

This equation describes an erection path of the n-bar module from $h = 0$, collapsed on the base plane, to the prescribed height $h < R$ on the hyper-surface (23). Any point connected from the current tensegrity configuration ξ_0 by a path on the hyper-surface (23) can be reached by satisfying self-equilibrium conditions. It is possible to deviate from a prescribed path as long as the new path remains on the hyper-surface (23). Fig. 6 illustrates the projected hyper-surface in the x, y, z-coordinate space of node 1 of the 3-bar tensegrity module by fixing nodes 2 and 3. At node 1, the y-axis is in the radial direction and the x-axis is tangent to the top inscribing circle. The forbidden grad Q direction is in the displacement direction of the mechanism, illustrated in Fig. 3a. It is noted that all continuous changes of configurations on the equilibrium hyper-surface require some work for changing reference lengths of elements including the work done against existing pre-stresses.

Eq. (23) guarantees the existence of admissible neighborhood of ξ_0 as well as the forbidden grad Q direction. By assuming that a small change $\Delta\xi_0$ has been made on the hyper-surface, the sensitivity of pre-stress and infinitesimal mechanism modes with a small configuration change or "geometrical imperfection" is investigated. Let the equilibrium matrix at ξ_0 be denoted by $\mathbf{A}_0$. The change of the configuration by

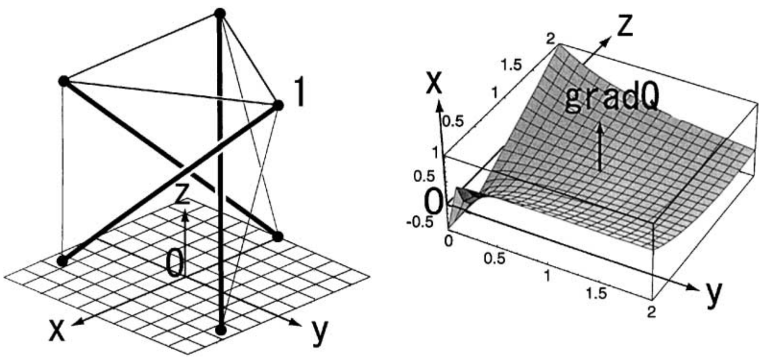

Fig. 6. An equilibrium hyper-surface of three-bar tensegrity module at node 1.

$\varepsilon = \|\Delta\boldsymbol{\xi}_0\|$ causes a small change of the equilibrium matrix from $\mathbf{A}_0$ to $\mathbf{A} = \mathbf{A}_0 + \varepsilon\mathbf{A}_1$. The resulting changes in $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ are both on the order of ε.

Consider a symmetric $n_{\mathrm{D}} \times n_{\mathrm{D}}$ matrix $\mathbf{C}_0$ with dim(null $\mathbf{C}_0$) = m. The eigenpairs are numbered in the order of "increasing" eigenvalues, and eigenvectors are orthonormalized:

$$\mathbf{C}_0\mathbf{d}_i = \mathbf{0}, \quad i = 1, \ldots, m, \tag{26a}$$

$$\mathbf{C}_0\mathbf{d}_i = \sigma_i^2\mathbf{d}_i, \quad i = m+1, \ldots, n_{\mathrm{D}}. \tag{26b}$$

Let a perturbed symmetric $n_{\mathrm{D}} \times n_{\mathrm{D}}$ matrix be $\mathbf{H} = \mathbf{C}_0 + \varepsilon\mathbf{C}_1$ with det $\mathbf{H} = 0$ for a small number ε. If m = dim(null $\mathbf{H}$) = dim(null $\mathbf{C}_0$) and nonzero eigenvalues are well separated from zero, it will be shown that the change of eigenvectors in null $\mathbf{H}$ is of the order of ε. The eigenvectors $\mathbf{h}_j$ corresponding to zero eigenvalues, i.e., $\mathbf{h}_j$ in null $\mathbf{H}$, satisfy

$$\mathbf{C}_0\mathbf{h}_j = -\varepsilon\mathbf{C}_1\mathbf{h}_j, \quad j = 1, \ldots, m. \tag{27a}$$

The right-hand term of Eq. (27a) can be expanded by $\mathbf{d}_{m+1}, \mathbf{d}_{m+2}, \ldots, \mathbf{d}_{n_{\mathrm{D}}}$ since it exists in the range space of $\mathbf{C}_0$:

$$\mathbf{C}_1\mathbf{h}_j = \sum_{i=m+1}^{n_{\mathrm{D}}} \langle \mathbf{d}_i, \mathbf{C}_1\mathbf{h}_j \rangle_{n_{\mathrm{D}}} \mathbf{d}_i. \tag{27b}$$

Next, $\mathbf{h}_j$ is expanded in $\mathbf{d}$'s as,

$$\mathbf{h}_j = \sum_{i=1}^{n_{\mathrm{D}}} a_i\mathbf{d}_i. \tag{28}$$

Substitution of Eq. (28) into Eq. (27a) with Eqs. (26a)–(27b) yields

$$\mathbf{h}_j = \mathbf{d}_j - \varepsilon \sum_{i=m+1}^{n_{\mathrm{D}}} \frac{1}{\sigma_i^2} \langle \mathbf{d}_i, \mathbf{C}_1\mathbf{h}_j \rangle_{n_{\mathrm{D}}}.$$

By approximating $\mathbf{h}_j$ on the right-hand side by $\mathbf{d}_j$, one finds a desired expansion:

$$\mathbf{h}_j \approx \mathbf{d}_j - \varepsilon \sum_{i=m+1}^{n_{\mathrm{D}}} \frac{1}{\sigma_i^2} \langle \mathbf{d}_i, \mathbf{C}_1\mathbf{d}_j \rangle_{n_{\mathrm{D}}}. \tag{29}$$

The above result applied to $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ shows that for a small change of configuration on the order of ε remaining on the self-equilibrium surface, the changes in pre-stress and infinitesimal mechanism modes are also on the order of ε if the first nonzero eigenvalue σ_{m+1}^2 is well separated from zero and dim(null $\mathbf{C}_0$) = dim(null $\mathbf{H}$). In the case of tensegrity structures, singular values are well separated from zero. Further, in the modal analysis of pre-stressed tensegrity structures illustrated in the first part of the paper, the natural frequencies corresponding to deformation modes are at least an order of magnitude larger than those of the infinitesimal mechanism modes. As Eqs. (21a)–(21e), (1.36b) and (1.36c) exhibit, the elastic modulus of infinitesimal mechanism modes is on the order of pre-stress, while that of the deformation modes is on the order of Young's modulus.

7. Concluding remarks

Linearized Lagrangian equations were applied to static analyses of tensegrity structures with $\mathrm{Mx} \leqslant 0$ With respect to the initial shape finding, the equivalence between the twist angle theorem based upon a simple geometrical analysis and the equilibrium analysis was shown for cyclic cylindrical tensegrity modules. It was found that in a pre-stressed tensegrity structure of first-order mechanism, all infinitesimal mechanisms were isotropically stiffened at each node by a single pre-stress mode. Both static and dynamic responses of tensegrity structures were characterized by infinitesimal mechanism modes. In the initial quasi-static loading, infinitesimal mechanisms exhibited soft response. As the deformation advances, the stiffness of tensegrity structures increased almost quadratically with the infinitesimal mechanism displacements. Therefore, if bars were properly designed against buckling, tensegrity structures exhibited stable hardening response. In summary, to analyze static and dynamic responses of pre-stressed tensegrity structures, it is extremely important to compute infinitesimal mechanisms of tensegrity modules and to interpret global deformation modes of tensegrity structures in terms of infinitesimal mechanism modes.

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